PHYS5110 — PLASMA PHYSICS

LECTURE 3 - PLASMA PROPERTIES: DEBYE SHIELDING AND ENERGY

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Plasma properties: Debye shielding

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1 DEBYE SHIELDING

We now consider a *negative* test charge Q immersed in a homogeneous plasma. Q will attract ions but repellants electrons. The displacement of electrons produces a *polarization charge*, which shields the plasma from the test charge. The theory of shielding has been developed first in 1923 by Peter Debye and Erich Hückel for dielectric fluids.

To derive the shielding potential ϕ for the charge Q we assume a homogeneous plasma with electrons of temperature T_e and density n_e and a fixed background of ions of density n_0. After the test charge has established equilibrium with the plasma its potential is given by the Poisson equation

$$\nabla^2\phi(r) = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\left(n_0 - n_e(r)\right) \text{ with } \phi(\infty) = 0. \tag{1}$$

In an electrostatic field the velocity distribution of the electrons is

$$f_e(\mathbf{v}) = n_0 \left\{\frac{m}{2\pi \mathrm{k_B} T}\right\}^{3/2} \exp\left\{-\frac{\frac{1}{2}m\mathbf{v}^2 + q\phi(r)}{\mathrm{k_B}T}\right\}.$$

The knowledge of $f_e(\mathbf{v})$ allows us to find the local electron number density $n_e(r)$

$$n_e(r) = \int_{\mathbb{R}} f_e(\mathbf{v})\,\mathrm{d}\mathbf{v} = n_0 \exp\left\{\frac{e\phi(r)}{\mathrm{k_B}T}\right\},$$

electrons: $q = -e$

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which we substitute into Eq. (1)

$$\nabla^2\phi = -\frac{e}{\epsilon_0}n_0\left(1-\exp\left\{\frac{e\phi}{\mathrm{k_B}T}\right\}\right).$$

We expand the exponential term into a Taylor series to linearize the quation for ϕ

$$\exp\left\{\frac{e\phi}{\mathrm{k_B}T}\right\} = 1+\frac{e\phi}{\mathrm{k_B}T}+\frac{1}{2}\left(\frac{e\phi}{\mathrm{k_B}T}\right)^2+\frac{1}{3!}\left(\frac{e\phi}{\mathrm{k_B}T}\right)^3+\cdots$$

and keep only the first two terms

$$\nabla^2\phi \approx \frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}.$$

Because the plasma is isotropic we now want to make use of the spherical symmetry of the problem. To this aim we express the Laplace operator in spherical coordinates

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right)+\frac{1}{r^2\sin^2\theta}\partial^2_\phi\phi$$

and drop the symmetric angular terms

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = \frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}.$$

This leads to an ordinary second order linear differential equation

$$\begin{aligned}\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T} &= 0\\ \frac{1}{r}\partial_r^2\left(r\phi\right)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T} &= 0\\ \partial_r^2\left(r\phi\right)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}\left(r\phi\right) = y''-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}y &= 0 \text{ with } y=\left(r\phi\right).\end{aligned}$$

The solutions of $y''+a^2y=0$ have the general form

$$y(x) = \frac{c}{x}\exp\left(\pm ax\right),$$

from which follows that

$$\phi(r) = \frac{A}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

with

$$\boxed{\lambda_D^2 = \frac{\epsilon_0\mathrm{k_B}T_e}{n_0e^2}} \tag{2}$$

being the *Debye length*. The value for the constant A can be found by using the fact that at large distances $\phi(r)$ must asymptotically approach *Coulomb's law* and we yield

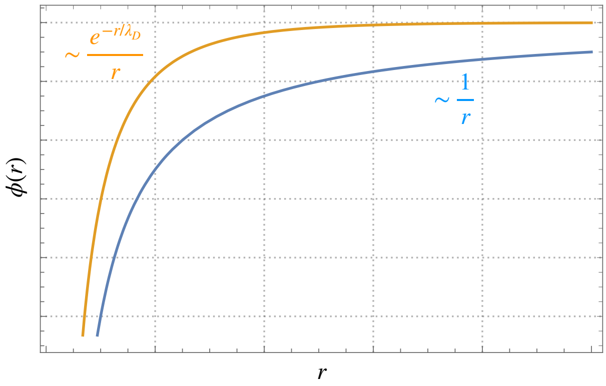

Figure 1: Comparison between the Debye-Hückel potential (orange) of a charge immersed in a plasma and the Coulomb potential (blue) of a free charge.

the so-called *Debye-Hückel potential*

$$\boxed{\phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right)} \tag{3}$$

(Fig. 1). A useful relation for the Debye length is

$$\lambda_D = 7430\mathrm{m}\sqrt{\frac{T}{\mathrm{eV}}\frac{\mathrm{m}^{-3}}{n}}. \tag{4}$$

2 ENERGY

Let us return to the *Debye-Hückel* potential

$$\phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{e^{-r/\lambda_D}}{r}, \tag{5}$$

which we have discussed in the last lecture. $\phi(r)$ describes the *effective* potential of an electron in a plasma, which is not any longer just $\sim r^{-1}$. This results from the necessity for a plasma to maintain charge neutrality by arranging the electrons in a shielding configuration. This means nothing else that the motion of particles in a plasma is not as random as in a neutral gas and that the energy of a plasma is not just the interior energy of a neutral gas. Following *Debye* and *Hückel* we will now determine the contribution of the correlated plasma particle motion to the energy.

The energy of a system of N electrostatically interacting charged particles is

$$E_e = \frac{N}{2}e\phi_a,$$

where ϕ_a is the potential of the field resulting by the other charged particles acting on the a^{th} particle. We can find ϕ_a by expanding Eq. 5 into a Taylor series and dropping the non-linear terms

$$\phi(r) \approx \frac{e}{4\pi\epsilon_0}\frac{1}{r} - \frac{e}{4\pi\epsilon_0}\frac{1}{\lambda_D}.$$

The first term is the *Coulomb* field of the particle itself, while the second term is the field resulting from the other particles, i.e. ϕ_a, and thus

$$E_e = -\frac{N}{8}\frac{e^2}{\pi\epsilon_0}\frac{1}{\lambda_D} = -\frac{N}{8}\frac{e^2}{\pi\epsilon_0}\left(\frac{n_0 e^2}{\epsilon_0 \mathrm{k_B} T_e}\right)^{1/2},$$

$$= -N\frac{e^3}{8\pi\epsilon_0^{3/2}} n_0^{1/2}\left(\frac{1}{\mathrm{k_B} T_e}\right)^{1/2}$$

$$= -N^{3/2}\frac{e^3}{8\pi\epsilon_0^{3/2}}\left(\frac{1}{V\mathrm{k_B} T_e}\right)^{1/2},$$

where V is the volume. This allows us now to compute the free energy F of the plasma by using

Free energy $F(T,V,N) = U - TS$

$$\frac{E}{T^2} = -\frac{\partial}{\partial T}\frac{F}{T},$$

and

$$F_{plasma} = F_{ideal} - T\int \frac{E_e}{T^2}\,\mathrm{d}T = F_{ideal} - \frac{2}{3}E_e.$$

From this follows that the pressure is

$$p(T,V,N) = -\frac{\partial F}{\partial V}|_{T,N} = \frac{N\mathrm{k_B}T}{V} - \frac{E_e}{3V}.$$